

THE NONEQUIVARIANT COHERENT-CONSTRUCTIBLE CORRESPONDENCE FOR TORIC SURFACES

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ABSTRACT. We prove the nonequivariant coherent-constructible correspondence conjectured by Fang-Liu-Treumann-Zaslow in the case of toric surfaces. Our proof is based on describing a semi-orthogonal decomposition of the constructible side under toric point blow-up and comparing it with Orlov's theorem.

1. INTRODUCTION

The nonequivariant coherent-constructible correspondence (NCCC) is a relation between the derived category of coherent sheaves on a toric variety and the derived category of constructible sheaves on a torus. NCCC is discovered by Bondal [Bon06] and formulated in terms of microlocal sheaf theory by Fang-Liu-Treumann-Zaslow [FLTZ11] as follows.

Let M be a free abelian group of finite rank and N be its dual free abelian group. Let further Σ be a smooth complete fan defined in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and X_{Σ} be the toric variety defined by Σ . We write the bounded derived category of coherent sheaves on X_{Σ} by $D^b(\text{coh } X_{\Sigma})$ and the bounded derived category of constructible sheaves on $M_{\mathbb{R}}/M$ by $D_c^b(M_{\mathbb{R}}/M)$. Here constructible sheaf means \mathbb{R} -constructible sheaf in the sense of [KS90, §8.4]. We define $\overline{\Lambda}_{\Sigma} \subset T^*M_{\mathbb{R}}/M$ as the coset of $\Lambda_{\Sigma} := \bigcup_{\sigma \in \Sigma} (\sigma^{\perp} + M) \times (-\sigma) \subset M_{\mathbb{R}} \times N_{\mathbb{R}} \cong T^*M_{\mathbb{R}}$. We write the full subcategory of $D_c^b(M_{\mathbb{R}}/M)$ spanned by objects whose microsupports are contained in $\overline{\Lambda}_{\Sigma}$ by $D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda}_{\Sigma})$. It is known that there exists a fully-faithful functor

$$\kappa_{\Sigma} : D^b(\text{coh } X_{\Sigma}) \hookrightarrow D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda}_{\Sigma})$$

which will be defined in (3.12).

Conjecture 1.1 (NCCC conjecture [FLTZ12, Tre10]). *The functor κ_{Σ} is an equivalence of triangulated categories*

$$D^b(\text{coh } X_{\Sigma}) \cong D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda}_{\Sigma}). \quad (1.1)$$

This conjecture is proved in special cases ([Tre10, SS16], see also Theorem 3.2). The equivariant version of this conjecture is called the coherent-constructible correspondence and proved by Fang-Liu-Treumann-Zaslow [FLTZ11].

In this paper, we prove Conjecture 1.1 in dimension 2:

Theorem 1.2. *Conjecture 1.1 holds for any 2-dimensional smooth complete fans.*

Our proof is based on Theorem 1.3 below.

Theorem 1.3. *Let Σ be an n -dimensional smooth complete fan and $\hat{\Sigma}$ be its blow-up at a torus fixed point. Then there exists a semi-orthogonal decomposition*

$$\mathrm{D}_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}}) \cong \left\langle \mathbb{D}(p_*\mathbb{C}_{(n-1) \cdot Z}), \dots, \mathbb{D}(p_*\mathbb{C}_Z), \mathrm{D}_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\Sigma}}) \right\rangle. \quad (1.2)$$

Here $p: M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/M$ is the quotient map, \mathbb{D} is the Verdier duality functor, and Z is a locally closed subset of $M_{\mathbb{R}}$ which will be defined in (4.1). This formula is an analogue of Orlov's theorem on the semi-orthogonal decomposition of derived category of coherent sheaves under blowing-up [Orl92, BO02]. In the situation of Theorem 1.3, Orlov's theorem says that the derived category of coherent sheaves on $X_{\hat{\Sigma}}$ has a semi-orthogonal decomposition

$$\mathrm{D}^b(\mathrm{coh} X_{\hat{\Sigma}}) \cong \left\langle \mathcal{O}_E((n-1)E), \dots, \mathcal{O}_E(E), \pi^* \mathrm{D}^b(\mathrm{coh} X_{\Sigma}) \right\rangle, \quad (1.3)$$

where $\pi: X_{\hat{\Sigma}} \rightarrow X_{\Sigma}$ is the blow-up morphism and E is the exceptional divisor. In Lemma 4.1, we will prove $\kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE)) \cong \mathbb{D}(p_*\mathbb{C}_{kZ})[-n]$. Then one can identify Theorem 1.3 with Orlov's theorem via NCCC. Comparing the semi-orthogonal components, we have the following:

Theorem 1.4. *Let Σ be an n -dimensional smooth complete fan and $\hat{\Sigma}$ be its blow-up at a torus fixed point. Conjecture 1.1 holds for Σ if and only if so is for $\hat{\Sigma}$.*

In the case of toric surfaces, toric MMP and Theorem 1.4 allow us to reduce Conjecture 1.1 to the case of $\mathbb{P}^1 \times \mathbb{P}^1$ which is already proved by Treumann [Tre10].

Nadler and Zaslow [NZ09, Nad09] identifies the derived Fukaya category of a cotangent bundle with the bounded derived category of constructible sheaves on its base space. By using this result, Fang-Liu-Treumann-Zaslow [FLTZ12] relates NCCC with homological mirror symmetry. Combining their results with our results, we obtain a version of homological mirror symmetry for toric surfaces:

Corollary 1.5. *Let Σ be a 2-dimensional smooth complete fan. Then there exists an equivalence of triangulated categories*

$$\mathrm{D}^b(\mathrm{coh} X_{\Sigma}) \cong \mathrm{DFuk}(T^*M_{\mathbb{R}}/M, \overline{\Lambda_{\Sigma}}). \quad (1.4)$$

The notations will be explained in Section 2.

This paper is organized as follows. In Section 2, we briefly recall microlocal sheaf theory of Kashiwara-Schapira [KS90]. In Section 3, we review an aspect of the NCCC and collect notations. In Section 4, we prove Theorem 1.3. Finally, in Section 5, we give a proof of Theorem 1.2 and 1.4.

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2. BACKGROUNDS FROM MICROLOCAL SHEAF THEORY

Let Y be a differentiable manifold and $D^b(\text{Sh } Y)$ be the derived category of \mathbb{C}_Y -module sheaves on Y . In this section, we always assume that $n \geq 2$.

Definition 2.1 ([KS90, Definition 5.1.2]). For $\mathcal{E} \in D^b(\text{Sh } Y)$, the *microsupport* $\text{SS}(\mathcal{E})$ of \mathcal{E} is a closed subset of T^*Y defined as follows; for $(x, \xi) \in T^*Y$, (x, ξ) is not contained in $\text{SS}(\mathcal{E})$ if there exists an open neighbourhood V of (x, ξ) such that for any C^1 function ψ with $\text{Graph}(d\psi) \subset V$ and

$$(\mathbb{R}\Gamma_{\{y|\psi(y) \geq \psi(x)\}} \mathcal{E})_x \simeq 0. \quad (2.1)$$

The microsupport detects the direction where the cohomology of the sheaf does not extend isomorphically.

For a cone $\gamma \subset N_{\mathbb{R}}$, we define the dual cone γ^\vee as

$$\gamma^\vee := \{m \in M_{\mathbb{R}} \mid \langle m, n \rangle \geq 0 \text{ for any } n \in \gamma\}. \quad (2.2)$$

For a subset $Z \subset Y$, we write the interior of Z by $\text{Int}(Z)$ and the relative interior of Z by $\text{Relint}(Z)$. We say a closed convex cone γ is *proper* (or strongly convex) if it satisfies $\gamma \cap (-\gamma) = \{0\}$.

Proposition 2.2 ([KS90, Proposition 5.1.1]). *For $\mathcal{E} \in D^b(\text{Sh } \mathbb{R}^n)$ and $(x_0, \xi_0) \in T^*\mathbb{R}^n$, $(x_0, \xi_0) \in \text{SS}(\mathcal{E})$ is equivalent to the following: For any neighbourhood V of x_0 , any positive integer ϵ , and any proper convex cone δ with $\xi_0 \in \text{Int}(\delta^\vee)$, there exists $x \in V$ such that*

$$\mathbb{R}\Gamma(H \cap (x - \delta), \mathcal{E}) \not\cong \mathbb{R}\Gamma(L \cap (x - \delta), \mathcal{E}) \quad (2.3)$$

where

$$H := \{y \in \mathbb{R}^n \mid \langle y - x_0, \xi_0 \rangle \geq -\epsilon\}, \quad (2.4)$$

$$L := \{y \in \mathbb{R}^n \mid \langle y - x_0, \xi_0 \rangle = -\epsilon\}. \quad (2.5)$$

Theorem 2.3 (the non-characteristic deformation lemma [KS90, Proposition 2.7.2]). *Let I be an open interval in \mathbb{R} , $\{U_s\}_{s \in I}$ be a family of open subsets in Y , and $\mathcal{E} \in D^b(\text{Sh } Y)$ satisfying the following:*

$$(1) \ U_s = \bigcup_{t < s} U_t \text{ for any } s \in I.$$

$$(2) \ U_t \setminus U_s \text{ is relatively compact for any } (s, t) \in I^2.$$

$$(3) \ (\mathbb{R}\Gamma_{Y \setminus U_t} \mathcal{E})_x \simeq 0 \text{ for any } s \leq t \text{ and } x \in \bigcap_{u > s} \text{Cl}(U_u \setminus U_s) \setminus U_t \text{ where Cl denotes taking closure.}$$

Then, we have $\mathbb{R}\Gamma(\bigcup_{t \in I} U_t, \mathcal{E}) \xrightarrow{\sim} \mathbb{R}\Gamma(U_s, \mathcal{E})$ for any $s \in I$.

Theorem 2.3 holds even if Y is not a manifold but Y is simply a Hausdorff topological space.

For a subset Z of Y , we define the *strict normal cone* $N_x Z \subset T_x Y$ of Z at $x \in Y$ as follows; the tangent vector $\xi \in T_x Y$ is not contained in $N_x Z$ if there exists a local coordinate U of x and two sequences $\{x_i\}_{i \in \mathbb{N}} \subset U \setminus Z$ and $\{y_i\}_{i \in \mathbb{N}} \subset U \cap Z$ satisfying convergence conditions; $x_i, y_i \rightarrow x$ and $x_i - y_i / |x_i - y_i| \rightarrow \xi / |\xi|$ in U . We define the *conormal cone* $N_x^* Z$ of Z at x as the dual cone $N_x Z^\vee$ of $N_x Z$.

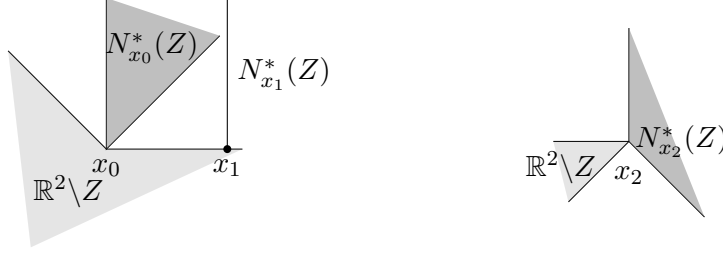


FIGURE 1. Some examples of conormal cone

Lemma 2.4. *Let γ be a closed convex cone in \mathbb{R}^n . Then we have*

- (1) $N_0^* \gamma = \gamma^\vee$, and
- (2) $N_0^*(\mathbb{R}^n \setminus \gamma) = -\gamma^\vee$.

Proof. These are clear from the definition of conormal cone. □

We say a subset Z of \mathbb{R}^n is *polyhedral* if it is defined by finite linear inequalities.

Lemma 2.5. *Let $Z, W \subset \mathbb{R}^n$ be a polyhedral subset. Then we have*

$$N_x^*(Z \cap W) = N_x^*Z + N_x^*W \quad (2.6)$$

for $x \in Z \cap W$.

Proof. This is also clear from the definition of conormal cone. □

Figure 1 shows some examples of Lemma 2.4.

Lemma 2.6 ([KS90, Corollary 5.4.9]). *We use the same notation as in the previous lemma. For $\mathcal{E} \in \text{D}^b(\text{Sh } Y)$, we assume that $\text{SS}(\mathcal{E}) \cap N_x^*Z \subset \{(x, 0)\}$. Then, we have $(\mathbb{R}\Gamma_Z(\mathcal{E}))_x \simeq 0$.*

Lemma 2.7. *Let $\gamma \subset \mathbb{R}^n$ be an n -dimensional closed polyhedral convex cone. We assume that there exists a proper n -dimensional closed polyhedral convex cone $\delta \subset \mathbb{R}^n$ such that $-\delta \cap \gamma = \{0\}$. For $\mathcal{E} \in \text{D}^b(\text{Sh } \mathbb{R}^n)$, we further assume that there exists a neighbourhood U of 0 such that $\text{SS}(\mathcal{E}) \cap U \times (\text{Relint}(\gamma^\vee) + \delta^\vee) = \emptyset$ under the canonical identification $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. Then, we have $(\mathbb{R}\Gamma_\gamma(\mathcal{E}))_0 \simeq 0$.*

Proof. We take a family of closed convex polyhedral cones $\{\gamma_s\}_{s \in [0, \infty)}$ having the following properties;

- (i) $\gamma \setminus \{0\} \subset \text{Int}(\gamma_s)$ for any $s \in [0, \infty)$,
- (ii) $\gamma_s \subset \gamma_t$ for $s \geq t$,
- (iii) $\bigcap_{s \in [0, \infty)} \gamma_s = \gamma$,
- (iv) $-\delta \cap \gamma_t = \{0\}$.

Take $x \in \text{Int}(\delta)$, then $0 \in (-\text{Int}(\delta) + cx) \cap \gamma_t$ for any $c \in (0, 1)$ and $t \in [0, \infty)$. The assumption (iv) tells us that $\{(-\delta + cx) \cap \gamma_t\}_{c \in (0, 1)}$ forms a neighbourhood system of 0 in γ_t . Hence we can take sufficiently small c such that $(-\delta + cx) \cap \gamma_t \subset U$. In the following, we rewrite cx as x .

We define

$$V_s := \begin{cases} \emptyset & \text{for } s < 0, \\ (\text{Int}(-\delta) + sx) \cap U & \text{for } s \in [0, 1]. \end{cases} \quad (2.7)$$

Fix some $s \in [0, 1]$ and $t \in [0, \infty)$ we define

$$U_u := V_{us} \cup (V_s \setminus \gamma_t) \quad (2.8)$$

for $u \in (-1, 1)$. Here $V_{us} = \emptyset$ for $us < 0$. Then we obtain a family of open subsets $\mathcal{U} := \{U_u\}_{u \in (-1, 1)}$. Since $(t - s) \cdot x \in \text{Int}(\delta)$ for $s < t \in [0, 1]$, we have $V_s \subset V_t$. Hence we have (1) of Theorem 2.3 for \mathcal{U} .

We also have (2) of Theorem 2.3 since $(-\delta + x) \cap \gamma_t \subset U$.

For any $a \leq b$, the set $\bigcap_{c>a} \text{Cl}(U_c \setminus U_a) \setminus U_b$ is empty or $\partial V_{us} \cap \gamma_t$. At $x \in \partial V_{us} \cap \gamma_t$, we have

$$\begin{aligned} N_x^*(\mathbb{R}^n \setminus U_u) &= N_x^*((\mathbb{R}^n \setminus V_{us}) \cap (\mathbb{R}^n \setminus (V_s \setminus \gamma_t))) \\ &= N_x^*(\mathbb{R}^n \setminus V_{us}) + N_x^*(\mathbb{R}^n \setminus (V_s \setminus \gamma_t)) \\ &= -N_x^*(V_{us}) + N_x^*(\gamma_t) \\ &\subset \delta^\vee + \gamma_t^\vee \end{aligned} \quad (2.9)$$

by Lemma 2.4 and Lemma 2.5. By the assumption (i), $\gamma_t^\vee \setminus \{0\}$ is contained in $\text{Relint}(\gamma^\vee)$ for $t \in [0, \infty)$. Hence, we have $U \times \gamma_t^\vee \cap \text{SS}(\mathcal{E}) \subset \{0\}$. Since $\partial V_s \cap \gamma_t \subset U$, $\text{SS}(\mathcal{E}) \cap U \times (\delta^\vee + \gamma_t^\vee) = \{0\}$. Then, by Lemma 2.6, $\{U_u\}_{u \in (-1, -1)}$ satisfy (3) of Theorem 2.3. Hence we can use Theorem 2.3 and we have

$$\mathbb{R}\Gamma(V_s, \mathcal{E}) \xrightarrow{\sim} \mathbb{R}\Gamma(V_s \setminus \gamma_t, \mathcal{E}). \quad (2.10)$$

In other words, we have $\mathbb{R}\Gamma_{\gamma_t}(V_s, \mathcal{E}) \simeq 0$ for any s and t .

Taking the limit with respect to t , we have $\mathbb{R}\Gamma_\gamma(V_s, \mathcal{E}) \simeq 0$ for any $s \in [0, 1]$. Since $-\delta \cap \gamma = \{0\}$, we know that $\{V_s \cap \gamma\}_{s \in [0, 1]}$ is an open neighbourhood system of 0 in γ . Finally, we have $(\mathbb{R}\Gamma_\gamma(\mathcal{E}))_0 \simeq 0$. \square

Let $E \cong \mathbb{R}^n$ be a vector space and $\gamma \subset E$ be an n -dimensional closed polyhedral convex cone. For a face τ of γ^\vee , we define a subset R_τ of E by

$$R_\tau := \bigcap_{\substack{\tau \prec \sigma \prec \gamma^\vee \\ \tau \neq \sigma}} (\tau^\perp \setminus \sigma^\perp) \quad (2.11)$$

where $\tau \prec \sigma$ means τ is a face of σ .

Lemma 2.8. *Let $\gamma \subset E \cong \mathbb{R}^n$ be an n -dimensional closed polyhedral convex cone. Let further δ be an n -dimensional proper polyhedral closed convex cone. For $\mathcal{E} \in \text{D}^b(\text{Sh } E)$, we assume the following:*

- (i) *There exists a neighbourhood V of 0 such that $(V \setminus \gamma) \times (\delta^\vee + \gamma^\vee) \cap \text{SS}(\mathcal{E}) \subset (V \setminus \gamma) \times \{0\} \cup \bigsqcup_{0 \neq \tau \prec \gamma^\vee} R_\tau \times \tau$.*
- (ii) *There exists $\xi_0 \in \text{Relint}(\gamma^\vee) \cap \text{Int}(\delta^\vee)$ such that $(0, \xi_0) \in \text{SS}(\mathcal{E})$. In particular, $\gamma \cap (-\delta) = \{0\}$.*
- (iii) *$(\gamma^\vee \setminus \text{Relint}(\gamma^\vee)) \cap \delta^\vee = \{0\}$.*

(iv) $\text{Int}(\gamma) \cap \text{Int}(\delta) \neq \emptyset$.

Then, for any neighbourhood U of 0, there exists $x \in U$ such that $\mathbb{R}\Gamma_\gamma(-\delta + x, \mathcal{E}) \neq 0$.

Proof. We note that (ii) and (iv) imply a family of closed subsets $\{(x - \delta) \cap \gamma \mid x \in \text{Int}(\gamma) \cap \text{Int}(\delta)\}$ forms a neighbourhood system of 0 in γ . Combining with $\xi_0 \in \text{Int}(\delta^\vee)$ from (ii), we can take a sufficiently small ϵ and $x_0 \in \text{Int}(\gamma) \cap \text{Int}(\delta)$ such that $\overline{W} := (x_0 - \delta) \cap H \subset V \cap U$ where

$$H := \{y \in V \mid \langle y, \xi_0 \rangle \geq -\epsilon\}. \quad (2.12)$$

We define an open neighbourhood W of 0 by $W := \text{Int}(\overline{W})$. Proposition 2.2 and the property $(0, \xi_0) \in \text{SS}(\mathcal{E})$ from (ii) ensures that there exists $x \in W$ such that

$$\mathbb{R}\Gamma(H \cap (x - \delta), \mathcal{E}) \xrightarrow{\neq} \mathbb{R}\Gamma(L \cap (x - \delta), \mathcal{E}) \quad (2.13)$$

where

$$L := \{y \in V \mid \langle y, \xi_0 \rangle = -\epsilon\}. \quad (2.14)$$

By the definition of W , it follows that $H \cap (x - \delta), L \cap (x - \delta) \subset V$.

Let us assume that $(x - \delta) \cap \gamma$ were empty. Then take $y_0 \in \text{Relint}((x - \delta \cap \gamma) \cap L)$. From $\xi_0 \in \text{Relint}(\gamma^\vee)$ of (ii), we can define a family of open subsets $\mathcal{V} := \{V_s\}_{s \in (-1, 1+\alpha)}$ of H with

$$V_s := (sx + (1 - s)y_0 - \text{Int}(\delta)) \cap H \text{ when } s \in (0, 1 + \alpha) \quad (2.15)$$

where $\alpha > 0$ is taken to be sufficiently small to satisfy $V_{1+\alpha} \subset V \setminus \gamma$. The family \mathcal{V} clearly satisfies (1) and (2) of Theorem 2.3. For $s \geq t$, the subset $\bigcap_{u>s} \text{Cl}(V_u \setminus V_s) \setminus V_t$ is \emptyset or $(sx + (1 - s)y_0 - \partial\delta) \cap H$. Hence the normal cone $N_y^*(V_{1+\alpha} \setminus V_s)$ is contained in $\xi_0 + \delta^\vee = \delta^\vee$ for $y \in \bigcap_{u>s} \text{Cl}(V_u \setminus V_s) \setminus V_t$. Since $V_{1+\alpha} \subset V \setminus \gamma$, (i) and (iii) tell us that $\text{SS}(\mathcal{E}) \cap N_y^*(V_{1+\alpha} \setminus V_s) = \{y\} \times \{0\}$ for $y \in \bigcap_{u>s} \text{Cl}(V_u \setminus V_s) \setminus V_t$. By Lemma 2.6, \mathcal{V} satisfies (3) of Theorem 2.3. Theorem 2.3 says that

$$\mathbb{R}\Gamma(V_{1+\alpha}, \mathcal{E}) \xrightarrow{\cong} \mathbb{R}\Gamma(L \cap ((1 + \alpha)x - \alpha y_0 - \delta), \mathcal{E}) \quad (2.16)$$

for any sufficiently small α . Taking $\alpha \rightarrow 0$, we have

$$\mathbb{R}\Gamma(H \cap (x - \delta), \mathcal{E}) \xrightarrow{\cong} \mathbb{R}\Gamma(L \cap (x - \delta), \mathcal{E}). \quad (2.17)$$

This contradicts to (2.13), hence we have $(x - \delta) \cap \gamma \neq \emptyset$.

We again define a family of open subsets $\mathcal{V}'_\alpha := \{V'_{s,\alpha}\}_{s \in (-1, 1)}$ of H with

$$V'_{s,\alpha} := ((s - 1)(1 + \alpha)x + (1 - s)y_0 + (((1 + \alpha)x - \text{Int}(\delta)) \setminus \gamma)) \cap H \text{ when } s \in (0, 1) \quad (2.18)$$

where α is taken to be sufficiently small to satisfy $V'_{1,\alpha} \subset V$. It is clear that the family \mathcal{V}'_α satisfies (1) and (2) of Theorem 2.3. For s , we define two sets:

$$Z_1 := ((s - 1)(1 + \alpha)x + (1 - s)y_0 + (((1 + \alpha)x - \partial\delta) \setminus \gamma)) \cap H. \quad (2.19)$$

$$Z_2 := (s - 1)(1 + \alpha)x + (1 - s)y_0 + (((1 + \alpha)x - \delta) \cap \partial\gamma). \quad (2.20)$$

Then, for $s \leq t$, $\bigcap_{u>s} \text{Cl}(V'_{u,\alpha} \setminus V'_{s,\alpha}) \setminus V'_{t,\alpha}$ is \emptyset or $Z_1 \sqcup Z_2$. If $y \in Z_1$, the normal cone $N_y^*(V'_{1,\alpha} \setminus V'_{s,\alpha})$ is contained in δ^\vee , hence we have $\text{SS}(\mathcal{E}) \cap N_y^*(V'_{1,\alpha} \setminus V'_{s,\alpha}) = \{y\} \times \{0\}$ by (i) and (iii). On the other hand, we have a decomposition $Z_2 = \bigsqcup_{\tau \prec \gamma^\vee} W_\tau$ where W_τ is defined by

$$W_\tau := (s-1)(1+\alpha)x + (1-s)y_0 + (((1+\alpha)x - \delta) \cap R_\tau \cap \gamma). \quad (2.21)$$

Since $y_0 - x \in -\text{Int}(\gamma)$, we have $\langle y_0 - x, \gamma^\vee \setminus \{0\} \rangle < 0$. Then we have $W_\tau \cap \bigcup_{\sigma \prec \gamma^\vee} R_\sigma \cup \bigcup_{\sigma \prec \tau} R_\sigma = \emptyset$. Moreover, if σ is not a face of τ and τ is not a face of σ , $\sigma \cap \tau$ is a proper face of σ and τ . For $y \in W_\tau$, we have

$$N_y^*(V'_{1,\alpha} \setminus V'_{s,\alpha}) = \{y\} \times (f(\delta^\vee) + \tau) \quad (2.22)$$

where $f(\delta^\vee)$ is one of the faces of δ^\vee . By (iii), we can see $\sigma \cap (\delta^\vee + \tau)$ is a proper face of σ . Taking intersection of $N_y^*(V'_{1,\alpha} \setminus V'_{s,\alpha})$ with the inclusion relation of (i), we have

$$\text{SS}(\mathcal{E}) \cap \text{Relint}(N_y^*(V'_{1,\alpha} \setminus V'_{s,\alpha})) = \emptyset. \quad (2.23)$$

By Lemma 2.7, the family \mathcal{V}'_α satisfies (3) of Theorem 2.3 and we have

$$\mathbb{R}\Gamma(H \cap V'_{1,\alpha}, \mathcal{E}) \xrightarrow{\cong} \mathbb{R}\Gamma(L \cap ((1+\alpha)x - \alpha y_0 - \delta), \mathcal{E}). \quad (2.24)$$

Take $\alpha \rightarrow 0$, then (2.24) becomes

$$\mathbb{R}\Gamma(H \cap (x - \delta) \setminus \gamma, \mathcal{E}) \xrightarrow{\cong} \mathbb{R}\Gamma(L \cap (x - \delta), \mathcal{E}). \quad (2.25)$$

Combining this with (2.13), we have

$$\mathbb{R}\Gamma(H \cap (x - \delta), \mathcal{E}) \xrightarrow{\cong} \mathbb{R}\Gamma(H \cap (x - \delta) \setminus \gamma, \mathcal{E}). \quad (2.26)$$

Hence we have $\mathbb{R}\Gamma_\gamma(H \cap (x - \delta), \mathcal{E}) \neq 0$. Since $\gamma \subset H$, we conclude $\mathbb{R}\Gamma_\gamma(x - \delta, \mathcal{E}) \neq 0$. \square

3. A REVIEW OF NCCC

Let M be a free abelian group of rank n and N be the dual free abelian group of M . We consider a smooth complete fan Σ defined in $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ and the associated toric variety X_Σ . For $\sigma \in \Sigma$, we write the corresponding affine toric subvariety of X_Σ by U_σ and the open immersion by $i_\sigma : U_\sigma \hookrightarrow X_\Sigma$. The *theta quasi-coherent sheaf* associated to $\sigma \in \Sigma$ is

$$\overline{\Theta}'(\sigma) := \mathcal{O}_\sigma := i_{\sigma*} \mathcal{O}_{U_\sigma} \in \text{D}^b(\text{Qcoh} X_\Sigma) \quad (3.1)$$

where $\text{D}^b(\text{Qcoh} X_\Sigma)$ is the bounded derived category of quasi-coherent sheaves. It is known that $\text{D}^b(\text{coh} X_\Sigma) \subset \langle \overline{\Theta}'(\sigma) \rangle_{\sigma \in \Sigma}$, where $\langle \cdot \rangle$ denotes the generated full subcategory [Tre10, Proposition 2.6].

We define the *theta quasi-constructible sheaf* associated to $\sigma \in \Sigma$ as

$$\overline{\Theta}(\sigma) := p_! \mathbb{D}(\mathbb{C}_{\sigma^\vee}) \in \text{D}_{qc}^b(M_\mathbb{R}/M) \quad (3.2)$$

where $p : M_\mathbb{R} \rightarrow M_\mathbb{R}/M$ is the quotient map, $\mathbb{D} : \text{D}_{qc}^b(M_\mathbb{R}) \rightarrow \text{D}_{qc}^b(M_\mathbb{R})^{op}$ is the Verdier duality functor, \mathbb{C}_{σ^\vee} is the zero-extension of the constant sheaf on σ^\vee , and $\text{D}_{qc}^b(M_\mathbb{R}/M)$ is the bounded derived category of quasi-constructible sheaves of \mathbb{C} -modules. Here, quasi-constructible (weakly

constructible in [KS90]) means that it is locally constant along some stratification but not necessarily of finite rank.

For $m \in \tau^\vee$, one can define $\theta'_m \in \text{Hom}_{\text{D}^b(\text{Qcoh} X_\Sigma)}^0(\overline{\Theta}'(\sigma), \overline{\Theta}'(\tau))$ the multiplication by the character χ^m ;

$$\theta'_m : \overline{\Theta}'(\sigma) \xrightarrow{\times \chi^m} \overline{\Theta}'(\tau). \quad (3.3)$$

This correspondence induces an isomorphism

$$\text{Hom}_{\text{D}^b(\text{Qcoh} X_\Sigma)}^i(\overline{\Theta}'(\sigma), \overline{\Theta}'(\tau)) \cong \begin{cases} \mathbb{C}[\tau^\vee \cap M] & \text{when } \sigma \supset \tau \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

Similarly, for $m \in \tau^\vee$, one can define $\theta_m \in \text{Hom}_{\text{D}_c^b(M_\mathbb{R}/M)}^0(\overline{\Theta}(\sigma), \overline{\Theta}(\tau))$ as the composition

$$\theta_m : \overline{\Theta}(\sigma) = p_! \mathbb{D}(\mathbb{C}_{\sigma^\vee}) = p_! \mathbb{D}(\mathbb{C}_{\sigma^\vee + m}) \xrightarrow{p_! \mathbb{D}(r_{\sigma^\vee + m}^{\tau^\vee})} \overline{\Theta}(\tau) \quad (3.5)$$

where χ^m is the character corresponding to m and $r_{\sigma^\vee + m}^{\tau^\vee} : \mathbb{C}_{\tau^\vee} \rightarrow \mathbb{C}_{\sigma^\vee + m}$ is the restriction map. This correspondence induces an isomorphism

$$\text{Hom}_{\text{D}_c^b(M_\mathbb{R}/M)}^i(\overline{\Theta}(\sigma), \overline{\Theta}(\tau)) \cong \begin{cases} \mathbb{C}[\tau^\vee \cap M] & \text{when } \sigma \supset \tau \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

The category $\Gamma(\overline{\Lambda}_\Sigma)$ is a dg-category whose set of objects is Σ and Hom-spaces are defined by

$$\text{hom}_{\Gamma(\overline{\Lambda}_\Sigma)}^i(\sigma, \tau) := \begin{cases} \mathbb{C}[\tau^\vee \cap M] & \text{when } \sigma \supset \tau \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.7)$$

with trivial differentials. We write the full sub dg-category of $\text{D}^{dg}(\text{Qcoh} X_\Sigma)$ (resp. $\text{D}_{qc}^{dg}(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma)$) spanned by $\{\overline{\Theta}'(\sigma)\}_{\sigma \in \Sigma}$ (resp. $\{\overline{\Theta}(\sigma)\}_{\sigma \in \Sigma}$) by $\overline{\Theta}'_{dg}$ (resp. $\overline{\Theta}_{dg}$). Then we have two quasi-equivalences of dg-categories

$$\Gamma(\overline{\Lambda}_\Sigma) \rightarrow \overline{\Theta}'_{dg}, \quad (3.8)$$

$$\Gamma(\overline{\Lambda}_\Sigma) \rightarrow \overline{\Theta}_{dg}. \quad (3.9)$$

Hence, we also have the quasi-equivalence of perfect dg-modules of dg-categories $\text{Per}_{dg} \overline{\Theta}'_{dg} \simeq \text{Per}_{dg} \overline{\Theta}_{dg}$. We write the equivalence induced on the homotopy categories by

$$K_\Sigma : \langle \overline{\Theta}' \rangle_{\sigma \in \Sigma} \xrightarrow{\cong} H^0(\text{Per}_{dg} \overline{\Theta}'_{dg}) \xrightarrow{\cong} H^0(\text{Per}_{dg} \overline{\Theta}_{dg}) \xrightarrow{\cong} \langle \overline{\Theta} \rangle_{\sigma \in \Sigma} \quad (3.10)$$

By the definition, K_Σ sends $\overline{\Theta}'(\sigma)$ to $\overline{\Theta}(\sigma)$ and θ'_m to θ_m [Tre10, Theorem 2.3].

To describe NCCC, we identify $M_\mathbb{R} \times N_\mathbb{R} \cong T^*M_\mathbb{R}$ and define

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma} (\sigma^\perp + M) \times (-\sigma) \subset T^*M_\mathbb{R} \quad (3.11)$$

and $\overline{\Lambda}_\Sigma \subset M_\mathbb{R}/M \times N_\mathbb{R} \cong T^*M_\mathbb{R}/M$ as the image of the Λ_Σ under the projection $\tilde{p} : T^*M_\mathbb{R} \rightarrow T^*M_\mathbb{R}/M$. We write the full subcategory of $\text{D}_c^b(M_\mathbb{R}/M)$ whose objects have microsupports in $\overline{\Lambda}_\Sigma$ by $\text{D}_c^b(X_\Sigma, \overline{\Lambda}_\Sigma)$.

Treumann showed that the essential image of $D^b(\text{coh } X_\Sigma)$ by K_Σ is contained in $D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma)$ [Tre10, Proposition 2.7]. Hence we obtain the following functor

$$\kappa_\Sigma := K_\Sigma|_{D^b(\text{coh } X_\Sigma)} : D^b(\text{coh } X_\Sigma) \rightarrow D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma). \quad (3.12)$$

The functor κ_Σ is fully-faithful, since it is a restriction of an equivalence.

Conjecture 1.1 is motivated by homological mirror symmetry. By the result of Nadler-Zaslow [NZ09] and Nadler [Nad09], the derived Fukaya category $\text{DFuk}(M_\mathbb{R}/M)$ of $M_\mathbb{R}/M$ and the bounded derived category of constructible sheaves $D_c^b(M_\mathbb{R}/M)$ on $M_\mathbb{R}/M$ are equivalent;

Theorem 3.1 (Nadler-Zaslow [NZ09], Nadler [Nad09]). *For a real analytic manifold X , there exists an equivalence of triangulated categories*

$$D_c^b(X) \cong \text{DFuk}(T^*X). \quad (3.13)$$

Fang-Liu-Treumann-Zaslow [FLTZ12] defined $D^b\text{Fuk}(T^*M_\mathbb{R}/M, \overline{\Lambda}_\Sigma)$ as the essential image of $D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma)$ under Nadler-Zaslow's equivalence. Combining Theorem 3.1 with Conjecture 1.1, we have a version of homological mirror symmetry for toric varieties.

There are some results on Conjecture 1.1. We say a smooth complete fan Σ is a zonotopal unimodular fan if Σ is obtained from a hyperplane arrangement and any linearly independent subset of the set of ray generators of Σ can be extended to \mathbb{Z} -basis of N . We say a smooth complete fan Σ is cragged when the following two conditions are satisfied:

- (1) For any subset S of Σ , the cone hull of S is a union of a subset of Σ .
- (2) For any linearly independent subset B of the set of ray generators R , the lattice generated by $\text{Cone}(B) \cap R$ has B as a \mathbb{Z} -basis.

Theorem 3.2 (Scherotzke-Sibilla [SS16], Treumann [Tre10]). *Let Σ is a smooth complete fan.*

- (1) *Conjecture 1.1 holds when Σ is zonotopal unimodular [Tre10, Corollary 4.5].*
- (2) *Conjecture 1.1 holds when Σ is cragged [SS16, Theorem 6.11].*

Both $D^b(\text{coh } X_\Sigma)$ and $D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma)$ carry monoidal structures as follows;

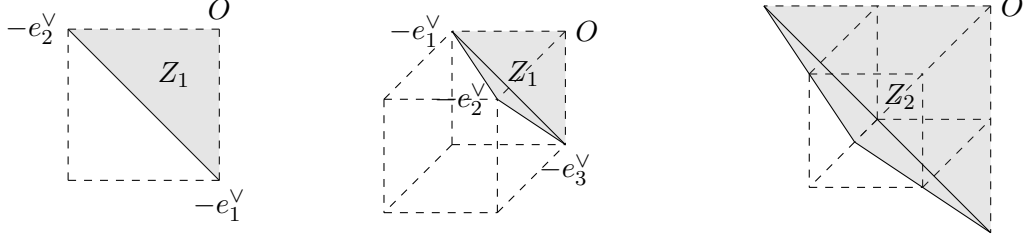
- (1) $\otimes := \boxtimes^\mathbb{L}$; derived tensor product in $D^b(\text{coh } X_\Sigma)$,
- (2) $\star := m_\dagger \circ \boxtimes^\mathbb{L}$; the composition of

$$\begin{aligned} D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma) \times D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma) \\ \xrightarrow{\boxtimes^\mathbb{L}} D_c^b(M_\mathbb{R}/M \times M_\mathbb{R}/M, \overline{\Lambda}_\Sigma \times \overline{\Lambda}_\Sigma) \\ \xrightarrow{m_\dagger} D_c^b(M_\mathbb{R}/M, \overline{\Lambda}_\Sigma) \end{aligned}$$

where $\boxtimes^\mathbb{L}$ is the exterior tensor product and m is the multiplication map with respect to the group structure of $M_\mathbb{R}/M$.

Some general properties of κ_Σ are known.

Theorem 3.3 (Fang-Liu-Treumann-Zaslow [FLTZ11], Treumann [Tre10]). *Let Σ be a smooth complete fan and $\hat{\Sigma}$ be a smooth complete subdivision of Σ . Let further $\pi^* : D^b(\text{coh } X_\Sigma) \rightarrow D^b(\text{coh } X_{\hat{\Sigma}})$*

FIGURE 2. Z_1 in $n = 2$, Z_1 and Z_2 in $n = 3$

be the pull-back along $\pi : X_{\hat{\Sigma}} \rightarrow X_{\Sigma}$ which is the morphism associated to the subdivision. Then we have the following:

- (1) There exists a natural equivalence $\kappa_{\hat{\Sigma}} \circ \pi^* \cong \iota \circ \kappa_{\Sigma}$ where ι is the inclusion functor $D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\Sigma}}) \rightarrow D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}})$ implied by $\overline{\Lambda_{\Sigma}} \subset \overline{\Lambda_{\hat{\Sigma}}}$.
- (2) The functor κ_{Σ} is monoidal with respect to the monoidal structures \otimes and \star .
- (3) There exists a natural equivalence, $\mathbb{D} \circ \kappa_{\Sigma} \cong \alpha^* \circ \kappa_{\Sigma} \circ D$ where $D := \mathbb{R}\mathcal{H}om(-, \mathcal{O}_{X_{\Sigma}})$ and $\alpha : M_{\mathbb{R}}/M \rightarrow M_{\mathbb{R}}/M$ is the inversion map $x \mapsto -x$.

4. NCCC AND BLOW-UP FORMULA

Let M be a free abelian group with $n := \text{rank} M \geq 2$, N be the dual of M , and Σ be a smooth complete fan defined in $N_{\mathbb{R}}$. We write a toric blow-up of X_{Σ} centered at a torus-fixed point by $\pi : X_{\hat{\Sigma}} \rightarrow X_{\Sigma}$, the exceptional divisor by $j : E \rightarrow X_{\Sigma}$, and the corresponding ray by $\rho_E \in \hat{\Sigma}$. We write the unique cone which corresponds to the affine toric subvariety containing the blow-up point by $\sigma_c \in \Sigma$. The cone σ_c is n -dimensional and we write edges of σ_c by ρ_1, \dots, ρ_n and the ray generator of ρ_i by e_i . Then, we have $e_E := \sum_{i=1}^n e_i$ for the ray generator of ρ_E . Since σ is smooth, $\{e_i\}_{i=1}^n$ forms a basis of N . The dual basis of M is denoted by $\{e_i^V\}_{i=1}^n$.

We define a locally closed subset $Z \subset M_{\mathbb{R}}$ by

$$Z := \{m \in M_{\mathbb{R}} \mid \langle m, e_E \rangle \geq -1\} \cap \{m \in M_{\mathbb{R}} \mid \langle m, e_i \rangle < 0 \text{ for any } i\}. \quad (4.1)$$

Also recall that the functor

$$\kappa_{\hat{\Sigma}} : D^b(\text{coh } X_{\Sigma}) \rightarrow D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}}). \quad (4.2)$$

To prove Theorem 1.3, we prepare the following lemma.

Lemma 4.1. *For $k \geq 1$,*

$$\kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE)) \simeq \mathbb{D}(p_* \mathbb{C}_{Z_k})[-n] \quad (4.3)$$

where $Z_k := k \cdot Z$ and $p : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/M$ is the quotient map.

Some examples of the subsets Z_k are depicted in Figure 2.

Proof. To calculate $\kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE))$, we first consider the following resolution;

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(E) \rightarrow \mathcal{O}_E(E) \rightarrow 0 \quad (4.4)$$

where we write $\mathcal{O}_{X_{\hat{\Sigma}}}$ by \mathcal{O} .

Next, we take Čech resolutions

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O} & \xrightarrow{\epsilon} & \mathcal{O}(E) & \longrightarrow & \mathcal{O}_E(E) \longrightarrow 0 \\
& & \downarrow & & \downarrow r_n & & \downarrow [r_n] \\
0 & \longrightarrow & \bigoplus_{\sigma \in \hat{\Sigma}(n)} \mathcal{O}_{U_\sigma} & \xrightarrow{\epsilon_n} & \bigoplus_{\sigma \in \hat{\Sigma}(n)} \mathcal{O}_{U_\sigma}(E) & \longrightarrow & \text{Coker}(\epsilon_n) \longrightarrow 0 \\
& & \downarrow & & \downarrow r_{n-1} & & \downarrow [r_{n-1}] \\
0 & \longrightarrow & \bigoplus_{\sigma \in \hat{\Sigma}(n-1)} \mathcal{O}_{U_\sigma} & \xrightarrow{\epsilon_{n-1}} & \bigoplus_{\sigma \in \hat{\Sigma}(n-1)} \mathcal{O}_{U_\sigma}(E) & \longrightarrow & \text{Coker}(\epsilon_{n-1}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots
\end{array} \tag{4.5}$$

Hereafter, we will ignore cones in $\hat{\Sigma}$ which do not contain ρ_E , since they are irrelevant for the calculation of $\mathcal{O}_E(E)$ in the above diagram.

By definition, $\mathcal{O}_{U_\sigma} = \overline{\Theta}'(\sigma)$. On each affine toric variety U_σ , we can take some e_j such that $\chi^{e_j^\vee} \cdot \mathcal{O}_{U_\sigma}(E) = \mathcal{O}_{U_\sigma}$ as a subsheaf of the constant sheaf of the function field. We fix such j and write $e_\sigma^\vee := e_j^\vee$. The exact triangle,

$$\overline{\Theta}'(\sigma) \xrightarrow{\chi^{e_\sigma^\vee}} \overline{\Theta}'(\sigma) \longrightarrow \text{Coker}(\chi^{e_\sigma^\vee}) \xrightarrow{[1]} \overline{\Theta}'(\sigma)[1] \tag{4.6}$$

in $D_{qc}^b(M_{\mathbb{R}}/M)$ is sent to

$$\overline{\Theta}(\sigma) \xrightarrow{\kappa_{\hat{\Sigma}}(\chi^{e_\sigma^\vee})} \overline{\Theta}(\sigma) \longrightarrow \kappa_{\hat{\Sigma}}(\text{Coker}(\chi^{e_\sigma^\vee})) \xrightarrow{[1]} \overline{\Theta}(\sigma)[1] \tag{4.7}$$

by $\kappa_{\hat{\Sigma}}$.

On the other hand, there exists the following exact triangle in $D_{qc}^b(M_{\mathbb{R}}/M)$

$$\mathbb{C}_{\sigma^\vee - e_\sigma^\vee} \longrightarrow \mathbb{C}_{\sigma^\vee} \xrightarrow{[1]} \mathbb{C}_{Z_\sigma}[1] \longrightarrow \mathbb{C}_{\sigma^\vee}[1] \tag{4.8}$$

where the left arrow is the restriction map and $Z_\sigma := (\sigma^\vee - e_\sigma^\vee) \setminus (\sigma^\vee)$. Note that Z_σ does not depend on the choice of e_σ^\vee . Applying $p_! \circ \mathbb{D}$ to this triangle, we have

$$\overline{\Theta}(\sigma) \xrightarrow{\kappa_{\hat{\Sigma}}(\chi^{e_\sigma^\vee})} \overline{\Theta}(\sigma) \longrightarrow p_! \mathbb{D}(\mathbb{C}_{Z_\sigma}) \xrightarrow{[1]} \overline{\Theta}(\sigma)[1]. \tag{4.9}$$

Comparing (4.9) with (4.7), we have $\kappa_{\hat{\Sigma}}(\text{Coker}(\chi^{e_\sigma^\vee})) \cong p_! \mathbb{D}(\mathbb{C}_{Z_\sigma})$.

We next calculate $\kappa_{\hat{\Sigma}}([r_i])$. For $\sigma \in \Sigma(i+1)$ and $\tau \in \Sigma(i)$, the restriction map $\mathcal{O}_\sigma(E) \rightarrow \mathcal{O}_\tau(E)$ is translated into $\overline{\Theta}'(\sigma) \xrightarrow{\chi^{e_\tau^\vee - e_\sigma^\vee}} \overline{\Theta}'(\tau)$. We define the sheaf $\hat{\mathbb{C}}_{\sigma^\vee} := \mathbb{C}_{\sigma^\vee - e_\sigma^\vee}$ on $M_{\mathbb{R}}$, then the restriction map $\hat{\mathbb{C}}_{\sigma^\vee} \rightarrow \hat{\mathbb{C}}_{\tau^\vee}$ is mapped to $\kappa_{\hat{\Sigma}}(\chi^{e_\tau^\vee - e_\sigma^\vee})$ by applying $p_! \mathbb{D}$. On the other hand, we can see from the definition of $\kappa_{\hat{\Sigma}}$ that the morphism induced on $\text{Coker}(\epsilon_i) \rightarrow \text{Coker}(\epsilon_{i-1})$ is mapped to the composition of the restriction map $\mathbb{C}_{Z_\sigma} \rightarrow \mathbb{C}_{Z_\tau}$ and $p_! \mathbb{D}$. By summing up the restriction maps

$\hat{\mathbb{C}}_{\sigma^\vee} \rightarrow \hat{\mathbb{C}}_{\tau^\vee}$ with Čech signs and applying $p_! \mathbb{D}$, we have $\kappa_{\hat{\Sigma}}(r_i)$. Hence, we conclude that $\kappa_{\hat{\Sigma}}([r_i])$ is obtained by summing up the restriction maps $\mathbb{C}_{Z_\sigma} \rightarrow \mathbb{C}_{Z_\tau}$ with Čech signs and applying $p_! \mathbb{D}$.

Note that $\hat{\mathbb{C}}_\sigma$ does not depend on a specific choice of e_σ . We can observe that $\{Z_\sigma\}_{\rho_E \subset \sigma \in \hat{\Sigma}(2)}$ forms a covering of $Z_{\rho_E} \setminus Z_1$. Moreover, $\{Z_\sigma\}_{\rho_E \subset \sigma \in \hat{\Sigma}}$ coincides with the Čech covering obtained from $\{Z_\sigma\}_{\rho_E \subset \sigma \in \hat{\Sigma}(2)}$. Hence, we have

$$\mathbb{C}_{Z_1} \simeq \left(\bigoplus_{\sigma \in \hat{\Sigma}(1)} \mathbb{C}_{Z_\sigma} \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in \hat{\Sigma}(n-1)} \mathbb{C}_{Z_\sigma} \rightarrow \bigoplus_{\sigma \in \hat{\Sigma}(n)} \mathbb{C}_{Z_\sigma} \right) \quad (4.10)$$

where the differentials in the RHS are the Čech differentials, and we regard the first term in the RHS is in degree 0.

To sum up, we have

$$\begin{aligned} & \mathbb{D}(\mathbb{C}_{p(Z_1)})[-n] \\ & \simeq p_! \mathbb{D}(\mathbb{C}_{Z_1})[-n] \\ & \simeq p_! \mathbb{D} \left(\bigoplus_{\sigma \in \hat{\Sigma}(1)} \mathbb{C}_{Z_\sigma} \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in \hat{\Sigma}(n-1)} \mathbb{C}_{Z_\sigma} \rightarrow \bigoplus_{\sigma \in \hat{\Sigma}(n)} \mathbb{C}_{Z_\sigma} \right) [-n] \\ & \simeq \kappa_{\hat{\Sigma}} \left(\bigoplus_{\sigma \in \hat{\Sigma}(n)} \text{Coker}(\chi^{e_\sigma^\vee})[n] \rightarrow \bigoplus_{\sigma \in \hat{\Sigma}(n-1)} \text{Coker}(\chi^{e_\sigma^\vee})[n] \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in \hat{\Sigma}(1)} \text{Coker}(\chi^{e_\sigma^\vee})[n] \right) [-n] \\ & \simeq \kappa_{\hat{\Sigma}}(\mathcal{O}_E(E)). \end{aligned} \quad (4.11)$$

By Theorem 3.3 (2) and (3), we have

$$\begin{aligned} \kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE)) & \simeq \mathbb{D}(\mathbb{C}_{p(Z_1)}) \star \cdots \star \mathbb{D}(\mathbb{C}_{p(Z_1)})[-n] \\ & \simeq \mathbb{D}(p_* \mathbb{C}_{Z_k})[-n]. \end{aligned} \quad (4.12)$$

This completes the proof. \square

Proof of Theorem 1.3. We note that $\kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE))$ is exceptional for $1 \leq k \leq n-1$, since $\kappa_{\hat{\Sigma}}$ is fully-faithful and $\mathcal{O}_E(kE)$ is exceptional. We write the triangulated hull of $\mathcal{O}_E(kE)$ by D_k . Then, $\kappa_{\hat{\Sigma}}(D_k)$ is an admissible full subcategory of $D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}})$. We have a semi-orthogonal decomposition [BK89],

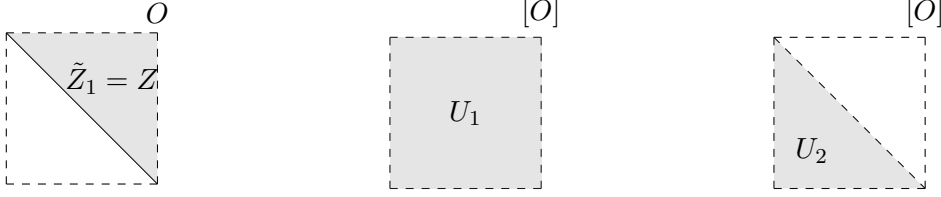
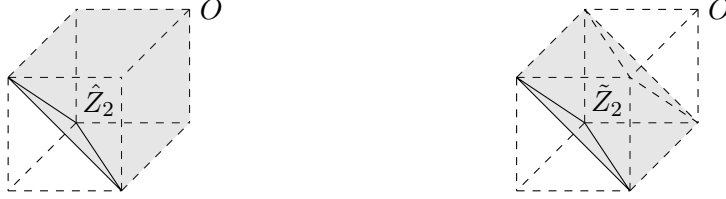
$$\begin{aligned} & D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}}) \\ & \cong \left\langle \kappa_{\hat{\Sigma}}(D_{-n+1}), \dots, \kappa_{\hat{\Sigma}}(D_{-1}), {}^\perp \left\langle \kappa_{\hat{\Sigma}}(D_{-n+1}), \dots, \kappa_{\hat{\Sigma}}(D_{-1}) \right\rangle \right\rangle. \end{aligned} \quad (4.13)$$

Hence, it is enough to show that ${}^\perp \left\langle \kappa_{\hat{\Sigma}}(D_{-n+1}), \dots, \kappa_{\hat{\Sigma}}(D_{-1}) \right\rangle = D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\Sigma}})$.

Step 1 (C). For $\mathcal{E} \in D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\Sigma}})$, we have

$$\mathbb{R}\text{Hom}(\mathcal{E}, \mathbb{D}(p_* \mathbb{C}_{Z_k})) \simeq \mathbb{R}\text{Hom}(p_* \mathbb{C}_{Z_k}, \mathbb{D}(\mathcal{E})). \quad (4.14)$$

We will show that this cohomology vanishes for $1 \leq k \leq n-1$.

FIGURE 3. \tilde{Z}_1 , U_1 , and U_2 in $n = 2$ FIGURE 4. \hat{Z}_2 and \tilde{Z}_2 in $n=3$ FIGURE 5. U_2 and U_3 in $n=3$

Let us first reduce the vanishing of (4.14) to (2) of Lemma 4.2. We define

$$F := \left\{ m = \sum_{i=1}^n a_i e_i^\vee \in M_{\mathbb{R}} \mid -1 \leq a_i < 0 \right\} \setminus M \quad (4.15)$$

and $\hat{Z}_k := Z_k \cap F$, then

$$Z_k \setminus \hat{Z}_k = \bigsqcup_{\substack{1 \leq j \leq k \\ l_1 + \dots + l_j = -k \\ I := \{i_1, \dots, i_j\} \subset \{1, \dots, n\}}} \left\{ m = \sum_{i=1}^n a_i e_i^\vee \in Z_k \mid \begin{array}{l} l_p \leq a_p < l_p + 1 \text{ for any } p \in I, \\ -1 \leq a_q < 0 \text{ for any } q \in \{1, \dots, k\} \setminus I \end{array} \right\}. \quad (4.16)$$

By induction, we only have to show

$$\mathbb{R}\mathrm{Hom}(p_* \mathbb{C}_{\hat{Z}_k}, \mathbb{D}(\mathcal{E})) \simeq 0 \quad (4.17)$$

to prove the vanishing of (4.14). Since there exists an exact triangle for $k \geq 2$

$$\mathbb{C}_{\tilde{Z}_k} \rightarrow \mathbb{C}_{\hat{Z}_k} \rightarrow \mathbb{C}_{Z_{k-1}} \xrightarrow{[1]} \mathbb{C}_{\tilde{Z}_k}[1] \quad (4.18)$$

where $\tilde{Z}_k := \hat{Z}_k - \hat{Z}_{k-1}$, we can further reduce the vanishing to

$$\mathbb{R}\mathrm{Hom}(p_* \mathbb{C}_{\tilde{Z}_k}, \mathbb{D}(\mathcal{E})) \simeq \mathbb{R}\Gamma_{p(\tilde{Z}_k)}(M_{\mathbb{R}}/M, \mathbb{D}\mathcal{E}) \simeq 0 \quad (4.19)$$

where we define $\tilde{Z}_1 := \hat{Z}_1$.

We define the subset U_k of $M_{\mathbb{R}}/M$ by

$$U_k := p(\tilde{U}_k) \quad (4.20)$$

$$\tilde{U}_k := F \setminus \hat{Z}_{k-1}, \quad (4.21)$$

then $p(\hat{Z}_k)$ is a closed subset of U_k (Figure 3, 4 and 5). Here we set $\hat{Z}_0 = \emptyset$. In the following, we fix an integer k with $1 \leq k \leq n-1$. Then, we have the following exact triangle:

$$\mathbb{R}\Gamma_{p(\hat{Z}_k)}(U_k, \mathbb{D}\mathcal{E}) \rightarrow \mathbb{R}\Gamma(U_k, \mathbb{D}\mathcal{E}) \rightarrow \mathbb{R}\Gamma(U_{k+1}, \mathbb{D}\mathcal{E}) \xrightarrow{[1]} \mathbb{R}\Gamma_{p(\hat{Z}_k)}(U_k, \mathbb{D}\mathcal{E})[1]. \quad (4.22)$$

Now the following lemma is clear:

Lemma 4.2. *For $\mathcal{E} \in \mathcal{D}_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\Sigma}})$, the following are equivalent:*

- (1) $\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathbb{D}(p_*\mathbb{C}_{Z_k})) \simeq 0$ for $1 \leq k \leq n-1$.
- (2) $\mathbb{R}\Gamma(U_k, \mathbb{D}\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\Gamma(U_{k+1}, \mathbb{D}\mathcal{E})$ for $1 \leq k \leq n-1$.

We will show the isomorphism (2) above in what follows. Let us define the family of open subsets $\mathcal{V} := \{V_s\}_{(-\infty, 1)}$ of U_k (Figure 6 and 7) by

$$V_s := \begin{cases} U_{k+1} & \text{for } s \in (-\infty, 0), \\ U_{k+1} \cup p\left(\left(\tilde{U}_k - \frac{1-s}{n} \cdot e_E\right) \cap F\right) & \text{for } s \in [0, 1). \end{cases} \quad (4.23)$$

Then, \mathcal{V} obviously satisfies (1) of Theorem 2.3.

For $s \leq t < 1$, we have

$$V_t \setminus V_s = \begin{cases} \emptyset \text{ or} \\ p\left(\left(\tilde{U}_k - \frac{1-s}{n} \cdot e_E\right) \cap \{n \in N_{\mathbb{R}} \mid \langle n, e_E \rangle \geq -k-t\} \cap F\right). \end{cases} \quad (4.24)$$

It follows that $V_t \setminus V_s$ is relatively compact in U_k i.e. \mathcal{V} satisfies (2) of Theorem 2.3.

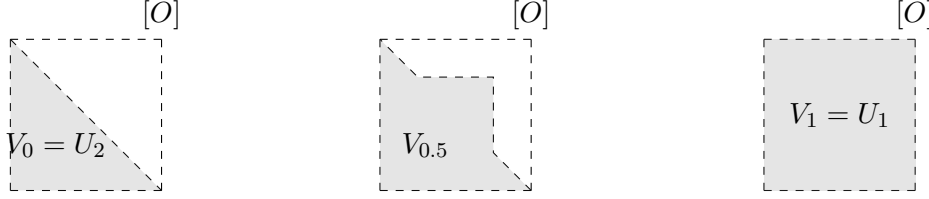
For $s \leq t$, we have

$$\bigcap_{u>s} \mathrm{Cl}(V_u \setminus V_s) \setminus V_t = \begin{cases} p\left(\left(\partial\tilde{U}_k - \frac{1-s}{n} \cdot e_E\right) \cap F\right) & \text{if } s = t > 0, \\ \emptyset & \text{otherwise.} \end{cases} \quad (4.25)$$

Take $y \in p\left(\left(\partial\tilde{U}_k - \frac{1-s}{n} \cdot e_E\right) \cap F\right)$. We can see that there exists a neighbourhood $B(y)$ of y such that $\gamma := N_y(U_k \setminus V_s)$ and $U_k \setminus V_s$ is canonically isomorphic in $B(y)$. The cone γ is an n -dimensional closed polyhedral convex cone which is contained in e_E^\vee under the canonical identification $T_y M_{\mathbb{R}}/M \cong M_{\mathbb{R}}$. Take an n -dimensional proper closed polyhedral convex cone $\delta \subset N_{\mathbb{R}}$ such that $\sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot e_i \setminus \{0\} \subset \mathrm{Int}(\delta)$ and $\delta \subset \mathrm{Int}(e_E^\vee)$. We also have $\gamma \subset e_E^\vee$. Hence we have $-\delta \cap \gamma = \{0\}$. We can also see that $\delta^\vee \setminus \{0\} \subset \sum_{i=1}^n \mathbb{R}_{>0} \cdot e_i^\vee$ and

$$\begin{aligned} \gamma^\vee &= N_y^*(U_k \setminus V_s) \subset N_y^*\left(p\left(\tilde{U}_k - \frac{1-s}{n} \cdot e_E\right)\right) + N_y^*(p(F)) \\ &\subset \sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot e_i. \end{aligned} \quad (4.26)$$

Hence we have $\mathrm{Relint}(\gamma^\vee) + \delta^\vee \subset \sum_{i=1}^n \mathbb{R}_{>0} \cdot e_i$.

FIGURE 6. V_0 , $V_{0.5}$, and V_1 in $n = 2$

Since $\text{SS}(\mathbb{D}\mathcal{E}) = -\text{SS}(\mathcal{E}) \subset -\overline{\Lambda_\Sigma}$, we have $U_k \times (\bigcup_{i=1}^n \mathbb{R}_{>0} e_i) \cap \text{SS}(\mathbb{D}\mathcal{E}) = \emptyset$. Hence, by Lemma 2.7, we obtain $(\mathbb{R}\Gamma_{U_k \setminus V_s}(\mathbb{D}\mathcal{E}))_y \simeq 0$. Then we can use Theorem 2.3 for \mathcal{V} and we know the middle arrow of (4.22) is an isomorphism. By Lemma 4.2, we conclude that $\mathcal{E} \in {}^\perp \langle \kappa_{\hat{\Sigma}}(\mathbb{D}_{-n+1}), \dots, \kappa_{\hat{\Sigma}}(\mathbb{D}_{-1}) \rangle$.

Step 2 (\supset). Conversely, take $\mathcal{E} \in {}^\perp \langle \kappa_{\hat{\Sigma}}(\mathbb{D}_{-r+1}), \dots, \kappa_{\hat{\Sigma}}(\mathbb{D}_{-1}) \rangle$. We have to show that $\text{SS}(\mathcal{E}) \subset \overline{\Lambda_\Sigma}$, or equivalently,

$$\text{SS}(\mathcal{E}) \cap \bigcup_{\substack{\rho_E \subset \sigma \in \hat{\Sigma} \\ \dim \sigma < n}} p\left(\text{Relint}\left((\sigma^\perp + M) \cap F\right)\right) \times (-\sigma) = \emptyset. \quad (4.27)$$

For a proof by contradiction, we assume that there exists an element $([x_0], -\xi_0)$ in the LHS of the above (4.27). Since $\text{SS}(\mathcal{E})$ is a conic Lagrangian subset [KS90], we can assume that $\xi_0 \in \text{Relint}(\sigma)$ for some $\sigma \in \hat{\Sigma}$ such that $\rho_E \subset \sigma$ and $\dim \sigma < n$. Fix $k \in \{1, \dots, n-1\}$, then we can take the unique lift $x_0 = \sum_{i=1}^n a_i e_i^\vee \in M_{\mathbb{R}}$ satisfying $-1 \leq a_i < 0$, $\sum_{i=1}^n a_i = -k$.

We can see that there exists a neighbourhood $B([x_0])$ of $[x_0]$ such that $\tilde{\gamma} := N_{[x_0]}(U_k \setminus U_{k+1})$ is canonically isomorphic to $U_k \setminus U_{k+1}$ in $B([x_0])$. Moreover, via the canonical identification $T_{[x_0]}M_{\mathbb{R}}/M \cong M_{\mathbb{R}}$, we have $\tilde{\gamma} \cong \sigma^\vee$. On the other hand, we can take an n -dimensional proper polyhedral closed convex cone $\tilde{\delta}$ such that $\sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot e_i^\vee \setminus \{0\} \subset \text{Int}(\tilde{\delta})$, $\tilde{\delta}^\vee \cap (\tilde{\gamma}^\vee \setminus \text{Relint}(\tilde{\gamma}^\vee)) = \{0\}$, and $\xi_0 \in \text{Int}(\tilde{\delta}^\vee)$.

The cones $\tilde{\gamma}$ and $\tilde{\delta}$ clearly satisfy (ii)-(iv) of Lemma 2.8. By the definitions of $\tilde{\gamma}$ and $\tilde{\delta}$, $(\tilde{\delta}^\vee + \tilde{\gamma}^\vee) \subset \sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot e_i^\vee$. Since $\text{SS}(\mathcal{E}) \subset \overline{\Lambda_\Sigma}$, we have

$$\begin{aligned} \text{SS}(\mathbb{D}\mathcal{E}) \cap (B([x_0]) \setminus \tilde{\gamma}) \times (\tilde{\delta}^\vee + \tilde{\gamma}^\vee) &\subset -\overline{\Lambda_\Sigma} \cap (B([x_0]) \setminus \tilde{\gamma}) \times \left(\sum_{i=1}^n \mathbb{R}_{\geq 0} e_i\right) \\ &\subset \bigcup_{0 \neq \tau \prec \sigma} \tau^\perp \times \tau \cup (B([x_0]) \setminus \tilde{\gamma}) \times \{0\} \\ &= \bigsqcup_{0 \neq \tau \prec \sigma = \tilde{\gamma}^\vee} R_\tau \times \tau \cup (B([x_0]) \setminus \tilde{\gamma}) \times \{0\}. \end{aligned} \quad (4.28)$$

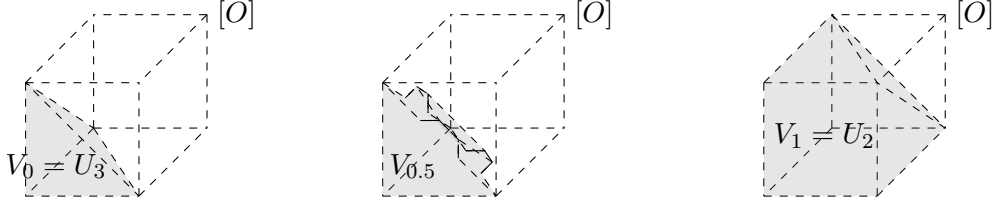
In the second line, we use the canonical identification $B([x_0]) \hookrightarrow T_{[x_0]}M_{\mathbb{R}}/M \cong M_{\mathbb{R}}$. This shows that $\tilde{\delta}$ and $\tilde{\gamma}$ also satisfy (i) of Lemma 2.8.

By Lemma 2.8, there exists x near x_0 such that

$$\mathbb{R}\Gamma_{U_k \setminus U_{k+1}}\left(p\left((x - \tilde{\delta}) \cap \tilde{U}_k \setminus \tilde{U}_{k+1}\right), \mathbb{D}\mathcal{E}\right) \neq 0. \quad (4.29)$$

We define $S := p\left(\tilde{U}_k \setminus \tilde{U}_{k+1} \cap (x - \tilde{\delta})\right)$. Then we have

$$\mathbb{R}\Gamma(U_{k+1} \cup S, \mathbb{D}\mathcal{E}) \neq \mathbb{R}\Gamma(U_{k+1}, \mathbb{D}\mathcal{E}). \quad (4.30)$$

FIGURE 7. V_0 , $V_{0.5}$ and V_1 in $n=3$

We define a family of open subsets $\mathcal{V}' := \{V'_s\}_{s \in (-\infty, 1)}$ in U_k (Figure 8) by

$$V'_s := \begin{cases} U_{k+1} \cup S & \text{for } s \in (-\infty, 0), \\ U_{k+1} \cup S \cup p\left(F \cap \left(s \cdot (\tilde{U}_k - x) + x\right)\right) & \text{for } s \in [0, 1). \end{cases} \quad (4.31)$$

We can see that the family \mathcal{V}' satisfies (1) and (2) of Theorem 2.3.

For $s \leq t$, we have

$$\bigcap_{u>s} \text{Cl}(V'_u \setminus V'_s) \setminus V'_t = \begin{cases} W_1 \cup W_2 \cup W_3 & s = t \geq 0, \\ \emptyset & \text{otherwise.} \end{cases} \quad (4.32)$$

where

$$W_1 := p\left(F \cap \left(s \cdot (\partial \tilde{U}_k - x) + x\right)\right) \setminus (\text{Cl}(U_{k+1}) \cup S), \quad (4.33)$$

$$W_2 := \partial S \cap p\left(F \cap \left(s \cdot (\partial \tilde{U}_k - x) + x\right)\right) \setminus W_3, \quad (4.34)$$

$$W_3 := p\left(\{m \in M_{\mathbb{R}} \mid \langle m, e_E \rangle = -k - 1\} \cap \left(s \cdot (\partial \tilde{U}_k - x) + x\right)\right). \quad (4.35)$$

Some examples of W_i are depicted in Figure 9. We take $y \in \bigcap_{u>s} \text{Cl}(V'_u \setminus V'_s) \setminus V'_s$ for $0 \leq s < 1$ and define a cone $\hat{\gamma}$ as $\hat{\gamma} := N_y(U_k \setminus V'_s)$. There exists a neighbourhood $B(y')$ of y' such that γ is canonically isomorphic to $U_k \setminus V'_s$ in $B(y')$.

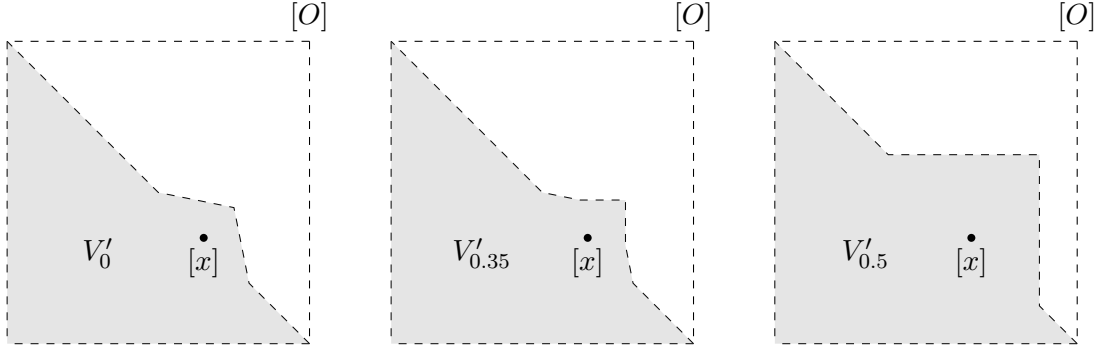
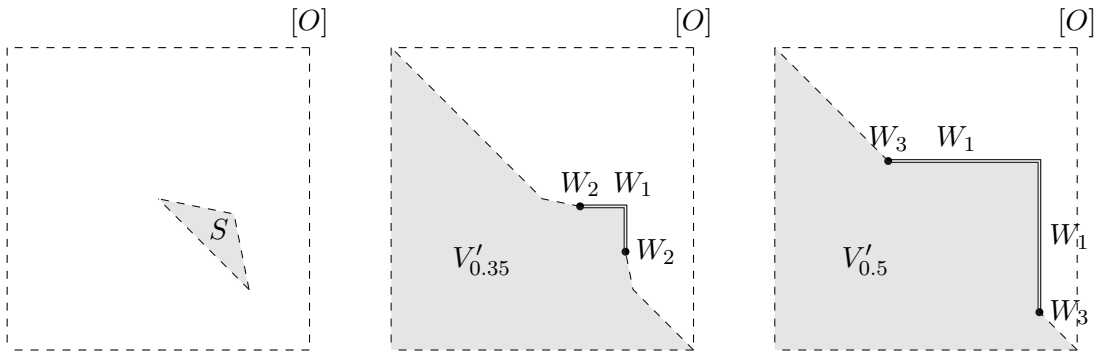
If $y \in W_2$, we have $\hat{\gamma}^\vee \subset \sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot e_i$ via the canonical identification $T_y^* M_{\mathbb{R}} / M \cong N_{\mathbb{R}}$. Since $\mathbb{D}\mathcal{E}$ does not have its microsupport in its first quadrant at y , we have $(\mathbb{R}\Gamma_{U_k \setminus V'_s}(\mathbb{D}\mathcal{E}))_y \simeq 0$ by Lemma 2.6.

If $y \in W_1$, we have $\hat{\gamma}^\vee \subset \sum_{i=0}^n \mathbb{R}_{\geq 0} \cdot e_i$. On the other hand, the microsupport of $\mathbb{D}\mathcal{E}$ at y is contained in a face of $\sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot e_i$. Hence we have $(\mathbb{R}\Gamma_{U_k \setminus V'_s}(\mathbb{D}\mathcal{E}))_y \simeq 0$ by Lemma 2.7.

If $y \in W_3$, $\hat{\gamma}$ is contained in some $\sigma \in \hat{\Sigma}$ such that $\rho_E \subset \sigma$. Hence, again by Lemma 2.7, we have $(\mathbb{R}\Gamma_{U_k \setminus V'_s}(\mathbb{D}\mathcal{E}))_y \simeq 0$.

Then, by using Theorem 2.3 for \mathcal{V}' , we have

$$\mathbb{R}\Gamma(U_k, \mathbb{D}\mathcal{E}) \simeq \mathbb{R}\Gamma(U_{k+1} \cup S, \mathbb{D}\mathcal{E}). \quad (4.36)$$

FIGURE 8. V'_0 , $V'_{0.35}$, and $V'_{0.5}$ in $n = 2$ FIGURE 9. S , W_i in $n = 2$

As a consequence, we have the following commutative diagram:

$$\begin{array}{ccc}
 \mathbb{R}\Gamma(U_{k+1} \cup S, \mathbb{D}\mathcal{E}) & \xrightarrow{\simeq} & \mathbb{R}\Gamma(U_{k+1}, \mathbb{D}\mathcal{E}) \\
 & \searrow \sim & \nearrow \sim \\
 & \mathbb{R}\Gamma(U_k, \mathbb{D}\mathcal{E}) &
 \end{array}
 \quad (4.37)$$

Since all arrows in (4.37) are restriction morphisms, this diagram is commutative. The upper arrow is (4.30) and the right arrow is the isomorphism (4.36). The left arrow is also an isomorphism by the assumption and Lemma 4.2. This diagram is obviously absurd. Hence, $([x_0], \xi_0) \notin \text{SS}(\mathcal{E})$. This completes the proof. \square

5. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1.4. First, suppose that $\kappa_{\hat{\Sigma}}: \text{D}^b(\text{coh } X_{\hat{\Sigma}}) \rightarrow \text{D}_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}})$ is an equivalence. Then, semi-orthogonal decomposition described in Orlov's theorem (1.3) implies

$${}^{\perp}\langle \kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE)) \rangle_{1 \leq k \leq n-1} \cong \kappa_{\hat{\Sigma}}\left(\pi^* \text{D}^b(\text{coh } X_{\Sigma})\right). \quad (5.1)$$

On the other hand, Theorem 1.3 implies

$${}^{\perp}\langle \kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE)) \rangle_{1 \leq k \leq n-1} \cong \iota\left(\text{D}_c^b(M_{\mathbb{R}}/M, \overline{\Lambda_{\hat{\Sigma}}})\right). \quad (5.2)$$

Hence,

$$\begin{aligned}
\kappa_{\Sigma}\left(D^b(\operatorname{coh} X_{\Sigma})\right) &\cong \kappa_{\hat{\Sigma}}\left(\pi^*D^b(\operatorname{coh} X_{\Sigma})\right) \\
&\cong {}^{\perp}\left\langle \kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE)) \right\rangle_{1 \leq k \leq n-1} \\
&\cong \iota\left(D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda}_{\Sigma})\right) \\
&\cong D_c^b(M_{\mathbb{R}}/M, \overline{\Lambda}_{\Sigma})
\end{aligned} \tag{5.3}$$

where the equivalence in the first line is Theorem 3.3 (1).

Conversely, we assume that Conjecture 1.1 holds for Σ . Take $\mathcal{E} \in \kappa_{\hat{\Sigma}}(D^b(\operatorname{coh} X_{\Sigma}))^{\perp}$, then $\mathcal{E} \in \kappa_{\hat{\Sigma}}(\mathcal{O}_E(kE))^{\perp}$ for $1 \leq k \leq n-1$ and $\mathcal{E} \in \kappa_{\hat{\Sigma}}(\pi^*D^b(\operatorname{coh} X_{\Sigma})) \cong \iota(D^b(M_{\mathbb{R}}/M, \overline{\Lambda}_{\Sigma}))^{\perp}$. Hence, by Theorem 1.3, $\mathcal{E} \cong 0$. This completes the proof. \square

Using Theorem 1.4, we have a proof of Conjecture 1.1 for smooth complete toric surfaces.

Proof of Theorem 1.2. We use toric minimal model program for toric surfaces. For any toric surfaces, after some toric blow-downs, we have \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or Hirzebruch surfaces \mathbb{F}_n . By Theorem 1.4, it is enough to show that Conjecture 1.1 holds for these toric minimal models. Moreover, \mathbb{P}^2 and \mathbb{F}_n are obtained by successive blow-ups and blow-downs of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, all cases may be reduced to the Conjecture 1.1 for $\mathbb{P}^1 \times \mathbb{P}^1$. The last case has already been shown by Treumann (Theorem 3.2 (1)). This completes the proof. \square

Finally, we obtain Corollary 1.5 by combining Theorem 1.2 with Theorem 3.1.

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